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Using Fractional Laplace Transform to Solve Fractional Differential Equations

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Abstract: **The purpose of this paper is to solve fractional differential equations by using fractional Laplace transforms based on Jumarie's modified Riemann-Liouville (R-L) fractional derivative. A new multiplication of fractional analytic functions plays a vital role in this research. We give some examples to illustrate the application of fractional Laplace transform in solving fractional differential equations. In fact, the method we use is a natural generalization of the Laplace transforms of analytic functions.**

Keywords: **fractional differential equations, fractional Laplace transforms, Jumarie's modified R-L fractional derivative, new multiplication, fractional analytic functions.**

I. INTRODUCTION

Fractional derivatives of non-integer orders are widely used in physics, mechanics, dynamics, and mathematical economics [1-8]. Until now, the rules of fractional derivative are not unique. Many authors have given the definition of fractional derivative. The commonly used definition is the Riemann-Liouvellie (R-L) definition [9-12]. Other useful definitions include Caputo definition of fractional derivative, Grunwald Letinikov (G-L) fractional derivative, conformable fractional derivative, and Jumarie's modified R-L fractional derivative [9-12]. Jumarie's modification of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function [13].

This paper introduces some fractional analytic functions such as the fractional exponential function, fractional cosine and sine functions. Based on Jumarie type of modified R-L fractional derivative, some fractional differential equations are solved by using the fractional Laplace transform of some fractional analytical functions. A new multiplication of fractional analytic functions plays an important role in this study. On the other hand, the method used in this paper is the extension of classical Laplace transform. In addition, the introduction and application of fractional Laplace transform can be referred to [14-16].

II. DEFINITIONS AND PROPERTIES

First, the fractional calculus used in this article is introduced below.

Definition 2.1: Suppose that α is a real number, and p is a positive integer. The Jumarie type of modified Riemann-Liouville fractional derivative [17] is defined by

$$\left({}_{t_0}D_t^{\alpha}\right)[f(t)] = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_{t_0}^{t}(t-\tau)^{-\alpha-1}f(\tau)d\tau, & \text{if } \alpha<0 \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{t_0}^{t}(t-\tau)^{-\alpha}[f(\tau)-f(a)]\,d\tau & \text{if } 0\le\alpha<1 \\ \frac{d^p}{dt^p}\left({}_{t_0}D_t^{\alpha-p}\right)[f(t)], & \text{if } p\le\alpha<p+1 \end{cases} \quad (1)$$

where $\Gamma(\ )$ is the gamma function. Moreover, we define the α-fractional integral of $f(t)$ by $\left({}_{t_0}I_t^{\alpha}\right)[f(t)] = \left({}_{t_0}D_t^{-\alpha}\right)[f(t)]$, where $\alpha>0$. If $\left({}_{t_0}I_t^{\alpha}\right)[f(t)]$ exists, then $f(t)$ is called an α-fractional integrable function. For any positive integer n, we define $\left({}_{t_0}I_t^{\alpha}\right)^n[f(t)] = \left({}_{t_0}I_t^{\alpha}\right)\left({}_{t_0}I_t^{\alpha}\right)\cdots\left({}_{t_0}I_t^{\alpha}\right)[f(t)]$, the n-th order fractional derivative of $f(t)$. We have the following properties.

Proposition 2.2: *If α, β, c are real numbers and $\beta \geq \alpha > 0$, then*

$$ {}_0D_t^{\alpha}\left[t^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, \tag{2}$$

and

$$ {}_0D_t^{\alpha}[c] = 0. \tag{3}$$

Next, we define the fractional analytic function.

Definition 2.3 ([18])**:** If , t_0 , and a_k are real numbers for all k, $t_0 \in (a,b)$, $0 < \alpha \leq 1$. Suppose that the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(t^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} (t-t_0)^{k\alpha}$ on some open interval $(t_0 - r, t_0 + r)$, then $f_\alpha(t^\alpha)$ is called α-fractional analytic at t_0, where r is the radius of convergence about t_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce some fractional analytic functions.

Definition 2.4 ([19, 20])**:** The Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \tag{4}$$

where α is a real number, $\alpha \geq 0$, and z is a complex number.

Definition 2.5 ([21])**:** Let $0 < \alpha \leq 1$, and p, t be real numbers. $E_\alpha(pt^\alpha) = \sum_{k=0}^{\infty} \frac{p^k t^{k\alpha}}{\Gamma(k\alpha+1)}$ is called α-fractional exponential function, and the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(pt^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k p^{2k} t^{2k\alpha}}{\Gamma(2k\alpha+1)}, \tag{5}$$

and

$$sin_\alpha(pt^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k p^{2k+1} t^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}, \tag{6}$$

In addition, the α-fractional hyperbolic cosine function and hyperbolic sine function are defined by

$$cosh_\alpha(pt^\alpha) = \frac{1}{2}[E_\alpha(pt^\alpha) + E_\alpha(-pt^\alpha)], \tag{7}$$

$$sinh_\alpha(pt^\alpha) = \frac{1}{2}[E_\alpha(pt^\alpha) - E_\alpha(-pt^\alpha)]. \tag{8}$$

Proposition 2.6 (fractional Euler's formula)**:** *Let $0 < \alpha \leq 1$, then*

$$E_\alpha(it^\alpha) = cos_\alpha(t^\alpha) + isin_\alpha(t^\alpha)\,. \tag{9}$$

A new multiplication of fractional analytic functions is introduced below.

Definition 2.7 ([18])**:** Assume that $0 < \alpha \leq 1$, $f_\alpha(t^\alpha)$ and $g_\alpha(t^\alpha)$ are two α-fractional analytic functions,

$$f_\alpha(t^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} t^{k\alpha}, \tag{10}$$

$$g_\alpha(t^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} t^{k\alpha}. \tag{11}$$

We define

$$\begin{aligned} & f_\alpha(t^\alpha) \otimes g_\alpha(t^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} t^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} t^{k\alpha} \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m \right) t^{k\alpha}. \end{aligned} \tag{12}$$

Theorem 2.8 (integration by parts for fractional calculus) ([22])**:** *If $0 < \alpha \leq 1$, a, b are real numbers, and $f_\alpha(t^\alpha), g_\alpha(t^\alpha)$ are α-fractional analytic functions, then*

$$\left({}_aI_b^\alpha\right)\left[f_\alpha(t^\alpha)\otimes\left({}_aD_t^\alpha\right)[g_\alpha(t^\alpha)]\right]=f_\alpha(t^\alpha)\otimes g_\alpha(t^\alpha)|_a^b-\left({}_aI_b^\alpha\right)\left[g_\alpha(t^\alpha)\otimes\left({}_aD_t^\alpha\right)[f_\alpha(t^\alpha)]\right]. \tag{13}$$

Proposition 2.9 ([21])**:** *If* $0<\alpha\le 1$, *and* λ,μ *are real number, then*

$$E_\alpha(\lambda t^\alpha)\otimes E_\alpha(\mu t^\alpha)=E_\alpha((\lambda+\mu)t^\alpha). \tag{14}$$

We give the definition of fractional Laplace transform as follows.

Definition 2.10: Assume that $0<\alpha\le 1$, s is a real variable, and $f_\alpha(t^\alpha)$ is an α-fractional analytic functions defined for all $t\ge 0$. The function $F_\alpha(s)$ defined by the α-fractional improper integral $\left({}_0I_{+\infty}^\alpha\right)[E_\alpha(-st^\alpha)\otimes f_\alpha(t^\alpha)]$ is called the α-fractional Laplace transform of the function f_α, and is denoted by $L_\alpha\{f_\alpha(t^\alpha)\}$. That is,

$$F_\alpha(s)=L_\alpha\{f_\alpha(t^\alpha)\}=\left({}_0I_{+\infty}^\alpha\right)[E_\alpha(-st^\alpha)\otimes f_\alpha(t^\alpha)]. \tag{15}$$

Furthermore, the given α-fractional analytic function $f_\alpha(t^\alpha)$ is called the inverse α-fractional Laplace transform of $F_\alpha(s)$ and is denoted by $L_\alpha^{-1}\{F_\alpha(s)\}$; that is, we shall write $f_\alpha(t^\alpha)=L_\alpha^{-1}\{F_\alpha(s)\}$.

Next, we introduce some properties of fractional Laplace transform.

Proposition 2.11 (linearity of fractional Laplace transform)**:** *The fractional Laplace transform is a linear operation; that is, for any fractional analytic functions* $f_\alpha(t^\alpha)$ *and* $g_\alpha(t^\alpha)$ *whose fractional Laplace transforms exist, then for any constants* a *and* b, *the fractional Laplace transform* $af_\alpha(t^\alpha)+b\,g_\alpha(t^\alpha)$ *exists and*

$$L_\alpha\{a\,f_\alpha(t^\alpha)+b\,g_\alpha(t^\alpha)\}=aL_\alpha\{f_\alpha(t^\alpha)\}+bL_\alpha\{g_\alpha(t^\alpha)\}. \tag{16}$$

Theorem 2.12 (first shifting theorem for fractional Laplace transform) ([14])**:** *Suppose that* $0<\alpha\le 1$, p,k *are real numbers, and* $f_\alpha(t^\alpha)$ *has the fractional Laplace transform* $F_\alpha(s)$ *for* $s>k$. *Then* $E_\alpha(-pt^\alpha)\otimes f_\alpha(t^\alpha)$ *has the fractional Laplace transform* $F_\alpha(s-p)$ *for* $s>k+p$. *In formula,*

$$L_\alpha\{E_\alpha(pt^\alpha)\otimes f_\alpha(t^\alpha)\}=F_\alpha(s-p). \tag{17}$$

Proposition 2.13 ([14])**:** *If* $0<\alpha\le 1$, s,t,p,ω *are real numbers,* $t\ge 0$, *and* n *is a positive integer. Then*

$$L_\alpha\{1\}=\frac{1}{s}, \text{ where } s>0 \tag{18}$$

$$L_\alpha\{E_\alpha(pt^\alpha)\}=\frac{1}{s-p}, \text{ where } s>p \tag{19}$$

$$L_\alpha\{t^{n\alpha}\}=\frac{\Gamma(n\alpha+1)}{s^{n+1}}, \text{ where } s>0 \tag{20}$$

$$L_\alpha\{cos_\alpha(\omega t^\alpha)\}=\frac{s}{s^2+\omega^2}, \text{ where } s>0 \tag{21}$$

$$L_\alpha\{sin_\alpha(\omega t^\alpha)\}=\frac{\omega}{s^2+\omega^2}, \text{ where } s>0 \tag{22}$$

$$L_\alpha\{cosh_\alpha(pt^\alpha)\}=\frac{s}{s^2-p^2}, \text{ where } s>|p| \tag{23}$$

$$L_\alpha\{sinh_\alpha(pt^\alpha)\}=\frac{p}{s^2-p^2}, \text{ where } s>|p| \tag{24}$$

$$L_\alpha\{E_\alpha(pt^\alpha)\otimes cos_\alpha(\omega t^\alpha)\}=\frac{s-p}{(s-p)^2+\omega^2}, \text{ where } s>p \tag{25}$$

$$L_\alpha\{E_\alpha(pt^\alpha)\otimes sin_\alpha(\omega t^\alpha)\}=\frac{\omega}{(s-p)^2+\omega^2}, \text{ where } s>p \tag{26}$$

Remark 2.14: *Let* $0<\alpha\le 1$, $f_\alpha(t^\alpha)$ *be an* α*-fractional analytic function. Then* $f_\alpha(t^\alpha)$ *has a Laplace transform if it does not grow too fast, that is, if for all* $t\ge 0$ *and some constants* M *and* k, *it satisfies the growth restriction*

$$\| f_\alpha(t^\alpha)\|\le ME_\alpha(kt^\alpha), \tag{27}$$

where $\| f_\alpha(t^\alpha)\|=[f_\alpha(t^\alpha)\otimes f_\alpha(t^\alpha)]^{\otimes\frac{1}{2}}$.

Theorem 2.15 (fractional Laplace transform of fractional derivatives)**:** *Let* $0<\alpha\le 1$, *and if* $f_\alpha(t^\alpha)$, $\left({}_0D_t^\alpha\right)[f_\alpha(t^\alpha)],\dots,$

$\left({}_0D_t^\alpha\right)^{n-1}[f_\alpha(t^\alpha)]$ are continuous for all $t \geq 0$ *and satisfy the growth restriction. Then the α-fractional Laplace transform of* $\left({}_0D_t^\alpha\right)^n[f_\alpha(t^\alpha)]$ *satisfies*

$$L_\alpha\left\{\left({}_0D_t^\alpha\right)^n[f_\alpha(t^\alpha)]\right\} = s^n L_\alpha\{f_\alpha(t^\alpha)\} - s^{n-1}f(0) - s^{n-2}\left({}_0D_t^\alpha\right)[f_\alpha(t^\alpha)](0) - \cdots - \left({}_0D_t^\alpha\right)^{n-1}[f_\alpha(t^\alpha)](0). \quad (28)$$

Proof Since $L_\alpha\{\left({}_0D_t^\alpha\right)[f_\alpha(t^\alpha)]\}$

$$= \left({}_0I_{+\infty}^\alpha\right)\left[E_\alpha(-st^\alpha)\otimes\left({}_0D_t^\alpha\right)[f_\alpha(t^\alpha)]\right]$$

$$= \lim_{t\to+\infty}\left({}_0I_t^\alpha\right)\left[E_\alpha(-st^\alpha)\otimes\left({}_0D_t^\alpha\right)[f_\alpha(t^\alpha)]\right]$$

$$= \lim_{t\to+\infty}[E_\alpha(-st^\alpha)\otimes f_\alpha(t^\alpha)] - f(0) + s\cdot\lim_{t\to+\infty}\left({}_0I_t^\alpha\right)[E_\alpha(-st^\alpha)\otimes f_\alpha(t^\alpha)]$$

(by integration by parts for fractional calculus)

$$= sL_\alpha\{f_\alpha(t^\alpha)\} - f(0).$$

Using induction yields the desired result holds.

Q.e.d.

III. SOME EXAMPLES

In the following, we give several examples to illustrate the application of fractional Laplace transform in solving fractional differential equations.

Example 3.1: If $0 < \alpha \leq 1$. Solve the second-order fractional differential equation

$$\left({}_0D_t^\alpha\right)^2[y_\alpha(t^\alpha)] - y_\alpha(t^\alpha) = t^\alpha,\quad y_\alpha(0) = 1,\ \left({}_0D_t^\alpha\right)[y_\alpha(t^\alpha)](0) = 1. \quad (29)$$

Solution Suppose that the α-fractional Laplace transform of $y_\alpha(t^\alpha)$ is $Y_\alpha(s)$. Then by Theorem 2.15,

$$s^2Y_\alpha(s) - sy_\alpha(0) - \left({}_0D_t^\alpha\right)[y_\alpha(t^\alpha)](0) - Y_\alpha(s) = \frac{\Gamma(\alpha+1)}{s^2}. \quad (30)$$

Therefore,

$$(s^2-1)Y_\alpha(s) = s + 1 + \frac{\Gamma(\alpha+1)}{s^2}. \quad (31)$$

Thus,

$$Y_\alpha(s) = \frac{s+1}{s^2-1} + \frac{\Gamma(\alpha+1)}{s^2(s^2-1)}$$
$$= \frac{1}{s-1} + \frac{\Gamma(\alpha+1)}{s^2-1} - \frac{\Gamma(\alpha+1)}{s^2}. \quad (32)$$

Hence,

$$y_\alpha(t^\alpha) = L_\alpha^{-1}\{Y_\alpha(s)\}$$
$$= L_\alpha^{-1}\left\{\frac{1}{s-1} + \frac{\Gamma(\alpha+1)}{s^2-1} - \frac{\Gamma(\alpha+1)}{s^2}\right\}$$
$$= L_\alpha^{-1}\left\{\frac{1}{s-1}\right\} + \Gamma(\alpha+1)\cdot L_\alpha^{-1}\left\{\frac{1}{s^2-1}\right\} - L_\alpha^{-1}\left\{\frac{\Gamma(\alpha+1)}{s^2}\right\}$$
$$= E_\alpha(t^\alpha) + \Gamma(\alpha+1)\cdot sinh_\alpha(t^\alpha) - t^\alpha. \quad (33)$$

Example 3.2: Let $0 < \alpha \leq 1$. Solve the following second-order fractional differential equation

$$\left({}_0D_t^\alpha\right)^2[y_\alpha(t^\alpha)] + 2\left({}_0D_t^\alpha\right)[y_\alpha(t^\alpha)] + 5y_\alpha(t^\alpha) = E_\alpha(-t^\alpha)\otimes sin_\alpha(t^\alpha),\quad y_\alpha(0) = 0,\ \left({}_0D_t^\alpha\right)[y_\alpha(t^\alpha)](0) = 1. \quad (34)$$

Solution If the α-fractional Laplace transform of $y_\alpha(t^\alpha)$ is $Y_\alpha(s)$, then

$$L_\alpha\left\{\left({}_0D_t^\alpha\right)^2[y_\alpha(t^\alpha)]\right\} + 2L_\alpha\{\left({}_0D_t^\alpha\right)[y_\alpha(t^\alpha)]\} + 5L_\alpha\{y_\alpha(t^\alpha)\} = L_\alpha\{E_\alpha(-t^\alpha)\otimes sin_\alpha(t^\alpha)\}. \quad (35)$$

Using Theorem 2.15 yields

$$\left\{s^2Y_\alpha(s)-sy_\alpha(0)-\left({}_0D_t^\alpha\right)[y_\alpha(t^\alpha)](0)\right\}+2\{sY_\alpha(s)-y_\alpha(0)\}+5Y_\alpha(s)=\frac{1}{(s+1)^2+1}. \quad (36)$$

$$(s^2+2s+5)Y_\alpha(s)-1=\frac{1}{s^2+2s+2}\,. \quad (37)$$

Therefore,

$$Y_\alpha(s)=\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}=\frac{1}{3}\cdot\frac{1}{(s+1)^2+1^2}+\frac{1}{3}\cdot\frac{2}{(s+1)^2+2^2}\,. \quad (38)$$

Thus,

$$\begin{aligned} y_\alpha(t^\alpha) &= L_\alpha^{-1}\{Y_\alpha(s)\} \\ &= L_\alpha^{-1}\left\{\frac{1}{3}\cdot\frac{1}{(s+1)^2+1^2}+\frac{1}{3}\cdot\frac{2}{(s+1)^2+2^2}\right\} \\ &= \frac{1}{3}\cdot L_\alpha^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\}+\frac{1}{3}\cdot L_\alpha^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\} \\ &= \frac{1}{3}\cdot E_\alpha(-t^\alpha)\otimes[sin_\alpha(t^\alpha)+sin_\alpha(2t^\alpha)]. \end{aligned} \quad (39)$$

Example 3.3: Assume that $0<\alpha\le 1$. Solve the third-order fractional differential equation

$$\left({}_0D_t^\alpha\right)^3[y_\alpha(t^\alpha)]-3\left({}_0D_t^\alpha\right)^2[y_\alpha(t^\alpha)]+3\left({}_0D_t^\alpha\right)[y_\alpha(t^\alpha)]-y_\alpha(t^\alpha)=t^{2\alpha}\otimes E_\alpha(t^\alpha),$$

$$y_\alpha(0)=1,\ \left({}_0D_t^\alpha\right)[y_\alpha(t^\alpha)](0)=0,\ \left({}_0D_t^\alpha\right)^2[y_\alpha(t^\alpha)](0)=-2\,. \quad (40)$$

Solution Let the α-fractional Laplace transform of $y_\alpha(t^\alpha)$ be $Y_\alpha(s)$. Since

$$L_\alpha\left\{\left({}_0D_t^\alpha\right)^3[y_\alpha(t^\alpha)]\right\}-3L_\alpha\left\{\left({}_0D_t^\alpha\right)^2[y_\alpha(t^\alpha)]\right\}+3L_\alpha\{\left({}_0D_t^\alpha\right)[y_\alpha(t^\alpha)]\}-L_\alpha\{y_\alpha(t^\alpha)\}=L_\alpha\{t^{2\alpha}\otimes E_\alpha(t^\alpha)\}. \quad (41)$$

It follows that

$$\left\{s^3Y_\alpha(s)-s^2y_\alpha(0)-s\left({}_0D_t^\alpha\right)[y_\alpha(t^\alpha)](0)-\left({}_0D_t^\alpha\right)^2[y_\alpha(t^\alpha)](0)\right\}-3\left\{s^2Y_\alpha(s)-sy_\alpha(0)-\left({}_0D_t^\alpha\right)[y_\alpha(t^\alpha)](0)\right\}$$

$$+3\{sY_\alpha(s)-y_\alpha(0)\}-Y_\alpha(s)=\frac{\Gamma(2\alpha+1)}{(s-1)^3}\,. \quad (42)$$

Therefore,

$$(s^3-3s^2+3s-1)Y_\alpha(s)-s^2+3s-1=\frac{\Gamma(2\alpha+1)}{(s-1)^3}\,. \quad (43)$$

And hence,

$$Y_\alpha(s)=\frac{s^2-3s+1}{(s-1)^3}+\frac{\Gamma(2\alpha+1)}{(s-1)^6}$$

$$=\frac{1}{s-1}-\frac{1}{(s-1)^2}-\frac{1}{(s-1)^3}+\frac{\Gamma(2\alpha+1)}{(s-1)^6}\,. \quad (44)$$

Thus,

$$\begin{aligned} y_\alpha(t^\alpha) &= L_\alpha^{-1}\{Y_\alpha(s)\} \\ &= L_\alpha^{-1}\left\{\frac{1}{s-1}-\frac{1}{(s-1)^2}-\frac{1}{(s-1)^3}+\frac{\Gamma(2\alpha+1)}{(s-1)^6}\right\} \\ &= L_\alpha^{-1}\left\{\frac{1}{s-1}\right\}-\frac{1}{\Gamma(\alpha+1)}\cdot L_\alpha^{-1}\left\{\frac{\Gamma(\alpha+1)}{(s-1)^2}\right\}-\frac{1}{\Gamma(2\alpha+1)}\cdot L_\alpha^{-1}\left\{\frac{\Gamma(2\alpha+1)}{(s-1)^3}\right\}+\frac{\Gamma(2\alpha+1)}{\Gamma(5\alpha+1)}\cdot L_\alpha^{-1}\left\{\frac{\Gamma(5\alpha+1)}{(s-1)^6}\right\} \\ &= E_\alpha(t^\alpha)-\frac{1}{\Gamma(\alpha+1)}\cdot t^\alpha\otimes E_\alpha(t^\alpha)-\frac{1}{\Gamma(2\alpha+1)}\cdot t^{2\alpha}\otimes E_\alpha(t^\alpha)+\frac{\Gamma(2\alpha+1)}{\Gamma(5\alpha+1)}\cdot t^{5\alpha}\otimes E_\alpha(t^\alpha) \end{aligned}$$

$$=\left[1-\frac{1}{\Gamma(\alpha+1)}\cdot t^\alpha-\frac{1}{\Gamma(2\alpha+1)}\cdot t^{2\alpha}+\frac{\Gamma(2\alpha+1)}{\Gamma(5\alpha+1)}\cdot t^{5\alpha}\right]\otimes E_\alpha(t^\alpha)\,. \quad (45)$$

IV. CONCLUSION

In this paper, fractional Laplace transform is used to solve fractional differential equations. In fact, the method we use is a generalization of the Laplace transform of analytic functions. In addition, the new multiplication we defined is a natural operation in fractional calculus and plays an important role in this article. In the future, we will study the problems of engineering mathematics and fractional calculus by using Jumarie type of modified R-L fractional derivative.

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